Indian Statistical Institute

Back-paper Examination 2022-2023

Analysis II, B.Math First Year

Time : 3 Hours Date : 05.06.2023 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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You may freely apply any of the theorems covered in class.

Q1: (20 marks) Define $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \cos^2 x & \text{if } x \in \mathbb{Q} \cap [0, \frac{\pi}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

Compute the upper and lower Riemann integrals of f over $[0, \frac{\pi}{2}]$. Discuss the Riemann integrability of f.

Q2: (20 marks) Determine the convergency of the following integrals:

$$(i) \int_1^\infty e^{-x^2}, \qquad (ii) \int_1^\infty \frac{\sin^2(x)}{x^2}.$$

Q3: (20 marks) Let $f : [1, \infty) \to [0, \infty)$ be a monotonically decreasing function. Prove that

$$\int_1^\infty f,$$

exists and is finite if and only if

$$\sum_{n=1}^\infty f(n).$$

exists and is finite.

Q4: (20 marks) Let $M > 0$. Prove that the series of functions

$$\sum_{n=1}^\infty \left(1 - \cos\frac{x}{n}\right),$$

converges uniformly on $(0, M)$.

Q5: (20 marks) Find the power series representation for the function

$$f(x) = \frac{x^2}{1 - x^3},$$

and determine the interval of convergence.